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Harold Weitzner

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plasma based on the

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Harold Weitzner

U. S. Department of Energy

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ABSTRACT

The cold plasma conductivity tensor is employed to explore the problem of wave propagation in a plasma without the use of the geometrical optics approximation or a plane stratified model. After the introduction of the appropriate electromagnetic potentials, the wave propagation problem reduces to a sixth order self-adjoint set of partial differential equations derivable from a self-adjoint variational principle. The usual lower and upper hybrid resonances as well as the plasma resonances can be identified as surfaces across which the system of equations changes from elliptic to partially elliptic and partially hyperbolic. In addition, the ion and electron cyclotron resonance surfaces, near which the cold plasma approximation breaks down, are also transition surfaces.

When the underlying plasma equilibrium is axisymmetric, although the waves need not be symmetric, a simpler set of potentials is given for which the wave propagation problem reduces to a fourth order system. The equations are again derivable from self-adjoint variational principles. The bulk of the paper considers the axisymmetric equilibrium case. For waves propagating at about the ion cyclotron frequency, it is proposed that the electric field is

C.1

continuous and its normal derivative is discontinuous across the ion cyclotron resonance surface. With this stipulation wave patterns and heating rates may be calculated in this frequency range. It is indicated how non-standard such a problem is. At frequencies on the order of the ion plasma frequency and including the lower hybrid resonance frequency a different situation pertains. First, a different geometric optics model is proposed in which certain spatial gradients of equilibrium quantities appear. The origin of the difference is the connection of the value of the small geometrical optics expansion parameter to the magnitude of various elements of conductivity tensor. The usual expansions assume that all the elements of the tensor are of order one in the geometrical optics parameter. Next, the equations are studied near the resonance surface. It is argued that if the resonance surfaces are not the same as flux surfaces then heating occurs only at the points of tangency of flux surfaces and the resonance surface. In this case the system is compared with the Tricomi equation of transonic flow. If the resonance surfaces and flux surfaces coincide then heating is possible on the entire resonance -- or flux -- surface. The great complexity of the general wave propagation and heating problem in this frequency range is described.

1. INTRODUCTION

The study of wave propagation originated decades ago with the examination of radio wave propagation in the ionosphere. Once the cold plasma conductivity tensor is developed, the problem reduces to a study of Maxwell's equations with the specified conductivity tensor. By far, the bulk of the work done in this field has been based on geometric optics or plane stratified medium approximations. Neither of these approximations is particularly valid in many typical laboratory plasma applications at moderately low frequencies and moderately long wavelengths. Additionally, the appearance of "resonances" and the associated plasma heating brings into question the relevance of the simple cold plasma model. Here, we intend to examine fairly generally the consequences of the cold plasma model for wave propagation problems. We intend to compare the situation in a plasma with the better known problems of scattering of waves in electromagnetic theory or in quantum mechanics. Our attack on this problem is based on an examination of the system of partial differential equations that defines the wave propagation problem. We will not be able to give definitive answers to many of the questions we raise, as the problems appear far beyond current knowledge. We will show that waves in plasmas behave, in many cases, quite unlike the much simpler version of waves in ordinary electromagnetic theory or in quantum mechanics. We believe that the questions we raise must be addressed directly to obtain reliable, quantitative measures of ion cyclotron or lower hybrid wave heating and model patterns in a plasma device. Any direct numerical calculations face immediately the problems posed here.

We employ Maxwell's equations with the cold plasma conductivity

tensor and we assume time harmonic dependence of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. As is well-known this set of partial differential equations is highly degenerate since the operator $\underset{\sim}{\nabla}\times$ has the large null space of all gradients. Thus, the usual prescriptions of how to determine the type of a system, elliptic hyperbolic, parabolic, or mixed, gives no information. We are unable even to give the order of the system. We show that the introduction of the usual electromagnetic potentials in an appropriate gauge generates a mathematically sensible system of equations, in fact, the system is self-adjoint and may be derived from a variational principle. The system, not surprisingly, can be identified as being sixth order. We can now determine the type of the system. Typically, time harmonic wave propagation or scattering problems are represented by elliptic systems. We find that for the cold plasma this is not necessarily the case. We find that the system always has some elliptic characteristics, but there are additionally transitions from elliptic to hyperbolic behavior at the plasma frequency resonance surface, lower and upper hybrid resonance surfaces, and at the fundamental ion and electron cyclotron resonance surfaces. The usual introduction of these resonances depends on properties of an infinite homogenous medium or possibly a stratified medium. We now see that the surfaces are related to an essential property of the system. The nature of the solution to wave propagation problems near such surfaces is one of the major topics of this work. We note that the cyclotron resonances are fundamentally different from the others. At cyclotron resonance elements of the conductivity tensor become unbounded, clearly indicating a failure of the cold plasma model, at the other resonances there is no obvious failure of the model. We

shall see that the solutions are fundamentally different near cyclotron resonance and lower hybrid resonance, as well. We believe that cyclotron resonance and lower hybrid resonance refer to fundamentally different phenomena.

The preceding results were obtained for the fully general, three-dimensional case. In many cases one is interested in wave propagation in a system in which the underlying equilibrium is axisymmetric although the waves lack this symmetry. For this situation we can give a better formulation of the wave propagation problem. We give two different representations, depending on whether the wave fields preserve the axisymmetry of the equilibrium or they destroy it. In both cases the electromagnetic fields are represented in terms of two potentials. The equations are of fourth order rather than sixth order as in the general three-dimensional case. We are unsure whether the change in order comes from a poor formulation in the three-dimensional case or partial integration of the system in the axisymmetric case. The characteristic surfaces for both of our fourth order systems are the same and correspond to two elliptic (imaginary) characteristics plus two others that may be either elliptic or hyperbolic. The surfaces of transition of type are not the same as in the three-dimensional case, although the upper and lower hybrid resonance transitions are unaffected. The systems of partial differential equations are derivable from self-adjoint variational principles.

The elements of the cold plasma conductivity tensor are very different in magnitude. In order to proceed further we must select a frequency regime and other physical parameters. For laboratory plasmas

we assume that the electron cyclotron and plasma frequencies are comparable and we examine separately waves at about the ion cyclotron frequency and the ion plasma frequency. The underlying small parameter to which we compare others is the electron-ion mass ratio. We consider first waves at about the ion cyclotron frequency. The situations in a mirror machine and a Tokamak are somewhat different and it is easier to describe them separately. In a mirror machine there is an elliptic-hyperbolic transition at the fundamental ion cyclotron resonance surface. The system is elliptic for frequencies above the ion cyclotron resonance frequency and hyperbolic below. Near the resonance surface the elements of the conductivity tensor are unbounded and the cold plasma model fails. In another paper (1) we have analyzed this situation with the help of a better model without singularities, the Vlasov equation, and we have shown that the solution to the wave propagation problem can be described in terms of the cold plasma model with the additional hypotheses that the electric field is continuous across the resonance surface, but that its normal derivative suffers a discontinuity. Additionally one can calculate the heating in terms of a simple surface integral. Thus, the solution to the wave propagation and energy absorption problem becomes the calculation of the electromagnetic fields for a system which in some regions is elliptic and in other regions is partly elliptic and partly hyperbolic. Across the transition surface the fields have specified discontinuities in their normal derivatives. This problem is quite non-standard, not only in the mixed type of the system, but also in the appearance of jump conditions. Presumably the self-adjoint variational principles will

allow direct calculation of the fields, but it should be emphasized that the problem is far from the ordinary.

In a Tokamak the ion cyclotron resonance problem is somewhat different. Formally, the system of equations exhibits no singularity at the ion cyclotron resonance although there is an elliptic-hyperbolic transition very close to the ion cyclotron resonance surface. The elliptic-hyperbolic transition occurs within the region in which the cold plasma model likely fails. Although we have not carried through the detailed analysis we believe that analysis similar to what has been done for the mirror machine would gives similar results. We expect the fields to be described by the cold plasma model except for jump conditions across the transition surface. It is interesting to note the appearance of the result that the component of the time varying electric field parallel to the steady magnetic field is small. In our treatment the result follows from the representation of the fields in terms of potentials and is not a property of the detailed solutions for the potentials. This result holds for both cases of mirrors and Tokamaks.

When we turn to the study of waves propagating at frequencies on the order of the ion plasma frequency and waves at lower hybrid frequencies we find a much more complex problem with many open questions. We consider in detail only axisymmetric modes in a Tokamak although the results holds for non-axisymmetric modes in a Tokamak as well. We have not explored the corresponding problems in a mirror. The elliptic-hyperbolic transition occurs exactly on the lower hybrid resonance surface and there are no other potential singularities of the system in this frequency range. Before we address the solutions near

the resonance surface we consider the usual geometric optics model approximation for the system. A direct examination of our system of equations shows that we obtain in the geometric optics limit a different set of equations from those one would find by the replacement of $\underset{\sim}{\nabla}$ by $i\underset{\sim}{k}$ in Maxwell's equations. Specifically, we find that spatial gradients of equilibrium quantities appear in our system. We found a similar phenomenon in a study with another author[2] of lower hybrid waves in a perpendicularly stratified medium. The origin of the difference is quite clear. In a physically reasonable modelling of a geometrical optics approximation to lower hybrid waves, the geometrical optics expansion parameter appears explicitly and non-trivially in the conductivity tensor. The usual geometrical optics treatments omit such dependence. Thus, even where geometrical optics is a good approximation we expect significantly different propagation characteristics from those usually given.

Finally we consider wave propagation near the lower hybrid resonance layer. We show first that the governing differential equations are very different depending on whether or not the plasma is perpendicularly stratified. If the medium is <u>not</u> perpendicularly stratified, the equations in normal form are essentially the Tricomi equation of transonic flow, where the resonance surface is just the sonic transition surface, and Laplace's equation. It has been well-established that the Tricomi equation with appropriate boundary data has smooth, well-behaved solutions.[3] Thus, we expect the wave propagation problem to have smooth solutions without any energy absorption across most of the resonance surface. Near those points at which the resonance surface is tangent to a flux surface, the system is

unlike the Tricomi equation and our argumentation fails, and we believe that energy absorption is possible. Similarity solutions for a model partial differential equation with the properties of lower hybrid waves show just these phenomena of no singularities or heating on flux surfaces or resonance surface except where they are parallel. When the plasma is perpendicularly stratified, the situation is quite different and we expect heating along the entire resonance -- or flux -- surface exactly as in our earlier work.(2) We cannot guarantee the surety of these results, but we believe they correspond to what is generally expected of such systems of equations. Even accepting the validity of our assertions we are not yet ready to formulate the energy absorption model near those points at which flux surfaces are tangent to resonance surfaces. The full solution of a wave propagation and energy absorption problem requires the resolution of this matter.

The organization of this paper follows the subject matter of this introduction. The next section shows how to formulate Maxwell's equations with the cold plasma conductivity tensor in a sufficiently explicit manner as to allow the determination of the mathematical type of the set of partial differential equations. The surfaces on which the type changes are found and shown to be the usual cyclotron, hybrid and plasma resonance surfaces. The third section shows how to introduce a simpler electromagnetic field representation in terms of two potentials for the case that the underlying plasma equilbrium is axisymmetric. The mathematical type of the system is also given as well as the surfaces of elliptic-hyperbolic transition. In preparation for subsequent material the scaling of the various small parameters in the electron to ion mass ratio is given. In the fourth section wave

propagation at about the ion cyclotron frequency is studied. The elliptic-hyperbolic transition is essentially the ion cyclotron resonance and the behavior near the resonance surface is described. The last section treats waves at about the ion plasma frequency. A modified geometrical optics which includes some spatial gradient effects is proposed. It is argued that plasma heating occurs only near those points at which the resonance surfaces are parallel to the flux surfaces.

II. SOME GENERAL PROPERTIES OF THE COLD PLASMA MODEL

In this section we give some standard material on Maxwell's equations in a dielectric medium and we explore some properties of the system. Our goal is to uncover enough information about the equations so that we may propose plausible methods for determining solutions without intervening slab or geometrical optics approximations. We would also like to make contact, where possible, with scattering and wave propagation problems in other areas of physics in order to validate any proposed solution methods. We start from the time harmonic variation form of Maxwell's equations

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -i\underset{\sim}{B} \tag{1}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = i\underset{\sim}{D} = i\underset{\approx}{\varepsilon}(\underset{\sim}{r},\omega)\cdot\underset{\sim}{E} \tag{2}$$

where $\underset{\approx}{\varepsilon}$ is the usual cold plasma dielectric tensor. In this form of Maxwell's equations we have assumed that $\underset{\approx}{r}$ is non-dimensional and measured in units of c/ω. For the cold plasma model $\underset{\approx}{\varepsilon}$ is hermitian and with coordinates in which the magnetic field is in the z direction

$$\underset{\approx}{\varepsilon} = \underset{\approx}{I} - \sum_{(j)} \begin{pmatrix} a_j & ib_j & 0 \\ -ib_j & a_j & 0 \\ 0 & 0 & c_j \end{pmatrix} \tag{3}$$

where

$$a_j = \omega_{p_j}^2/(\omega^2 - \Omega_{c_j}^2)$$

$$b_j = -a_j\Omega_{c_j}/\omega$$

$$c_j = \omega_{p_j}^2/\omega^2 ,$$

and where $\omega_{p_j}(\underset{\sim}{r})$ is the plasma frequency of the $j^{\underline{th}}$ species and $\Omega_{c_j}(\underset{\sim}{r})$ is the corresponding cyclotron frequency. We could easily include a tensor permeability as well as tensor dielectric, but no new ideas are involved and there is little relevance to plasma physics.

Despite the solid physical basis for the system (1),(2) it does have certain mathematical difficulties that are most obvious if we eliminate $\underset{\sim}{B}$ to obtain

$$\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \times \underset{\sim}{E}) = \underset{\sim}{\underset{\sim}{\varepsilon}}(\underset{\sim}{r},\omega)\cdot\underset{\sim}{E} . \tag{4}$$

One of the most elementary properties of a set of differential equations is its type: elliptic, hyperbolic, parabolic, or mixed. The type is usually associated with solution methods for the system. We may obtain the type by looking at the characteristic polynomial of the system. We keep only the highest derivatives, replace $\underset{\sim}{\nabla}$ by $\underset{\sim}{k}$ and the variable $\underset{\sim}{E}$ by $\delta\underset{\sim}{E}$. We interpret $\underset{\sim}{k}$ as the normal vector to some surface, $\delta\underset{\sim}{E}$ as the normal gradient to the surface and we consider the solvability for $\delta\underset{\sim}{E}$.[4] In this case the system is solvable if and only if

$$C_o = \det \begin{pmatrix} -(k_y^2+k_z^2) & k_x k_y & k_x k_z \\ k_x k_y & -(k_x^2+k_z^2) & k_y k_z \\ k_x k_z & k_y k_z & -(k_x^2+k_y^2) \end{pmatrix} \tag{5}$$

is not zero. Additionally, C_o is the symbol of the operator, and its properties describe the basic nature of the differential equations, for instance the degree of C_o is the order of the differential equations. An elementary computation shows

$$C_o \equiv 0 \ .$$

thus, we are unable to identify the type, or even the order of the system (1),(2). Obviously, the system is highly degenerate and the source of the degeneracy is clear: the operator $\underset{\sim}{\nabla}\times$ applied to any gradient vanishes. Thus, $\underset{\sim}{\nabla}\times$ has a large null space, and this null space changes the character of the system.

The introduction of the usual electromagnetic potentials resolves the degeneracy and permits direct analysis. We exhibit two possible sets of electromagnetic potentials with slightly different properties. In general we may introduce a vector potential $\underset{\sim}{A}$ and a scalar potential ϕ so that

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \times \underset{\sim}{A} \tag{6}$$

$$\underset{\sim}{E} = -i\underset{\sim}{A} - \underset{\sim}{\nabla}\phi \tag{7}$$

and (1) is identically satisfied while (2) becomes

$$\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{A}) - \Delta\underset{\sim}{A} = \underset{\sim}{\varepsilon}\cdot\underset{\sim}{A} - i\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\nabla}\phi \ ,$$

the governing differential equation. As is well known, the fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are invariant under the gauge transformation

$$\underset{\sim}{A} = \underset{\sim}{A}' + \underset{\sim}{\nabla}\lambda$$

$$\phi = \phi' - i\lambda \ .$$

Thus, we expect to adjoin one constraint on $\underset{\sim}{A}$ or ϕ in order to specify the potentials uniquely.

In the first set of potentials, $\underset{\sim}{A}_1$ and ϕ_1, we select radiation gauge

$$i\underset{\sim}{\nabla}\cdot\underset{\sim}{A}_1 = 0 \ . \tag{8}$$

so that our system is (8) plus

$$\Delta\underset{\sim}{A}_1 - i\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\nabla}\phi_1 = -\underset{\sim}{\varepsilon}\cdot\underset{\sim}{A}_1 \ . \tag{9}$$

If we now repeat the procedure for the construction of the characteristic form of the system we find

$$C_1 = \begin{vmatrix} k^2 & 0 & 0 & -i(\underset{\sim}{\varepsilon}\cdot\underset{\sim}{k})_x \\ 0 & k^2 & 0 & -i(\underset{\sim}{\varepsilon}\cdot k)_y \\ 0 & 0 & k^2 & -i(\underset{\sim}{\varepsilon}\cdot\underset{\sim}{k})_z \\ ik_x & ik_y & ik_z & 0 \end{vmatrix} \tag{10}$$

or

$$C_1 = -k^4 \underset{\sim}{k} \cdot \underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{k} \ . \tag{11}$$

We note that C_1 is not identically zero, and that it is of sixth degree in $\underset{\sim}{k}$ corresponding to the reasonable conclusion that (8), (9), or (1), (2), is a sixth order system. We examine the consequences of (11) shortly after we introduce the second set of potentials. The system (8), (9) is reasonable, but we see from (10) that the system is not symmetric or self-adjoint.

We may formulate the system in a self-adjoint manner if we select a different gauge. We introduce $\underset{\sim}{A}_2$ and ϕ_2 such that

$$i\nabla \cdot (\underset{\sim}{\underset{\sim}{\varepsilon}}^{+} \cdot \underset{\sim}{A}_2) = 0 \ , \tag{12}$$

where $\underset{\sim}{\underset{\sim}{\varepsilon}}^{+}$ is the adjoint of $\underset{\sim}{\underset{\sim}{\varepsilon}}$, $(\varepsilon^{+})_{ij} = \varepsilon^{*}_{ji}$. The remaining differential equation is

$$-\underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_2) + \Delta \underset{\sim}{A}_2 - i\underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{\nabla}\phi_2 = -\underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{A}_2 \ . \tag{13}$$

For these potentials the characteristic form is

$$C_2 = \begin{vmatrix} k^2-k_x^2 & -k_xk_y & -k_xk_z & -i(\underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{k})_x \\ -k_xk_y & k^2-k_y^2 & -k_yk_z & -i(\underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{k})_y \\ -k_xk_z & -k_yk_z & k^2-k_z^2 & -i(\underset{\sim}{\underset{\sim}{\varepsilon}} \cdot \underset{\sim}{k})_z \\ i(\underset{\sim}{k} \cdot \underset{\sim}{\underset{\sim}{\varepsilon}}^{+})_x & i(\underset{\sim}{k} \cdot \underset{\sim}{\underset{\sim}{\varepsilon}}^{+})_y & i(\underset{\sim}{k} \cdot \underset{\sim}{\underset{\sim}{\varepsilon}}^{+})_z & 0 \end{vmatrix} \tag{14}$$

so that

$$C_2 = -k^2(\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}\cdot\underset{\sim}{k})(\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}^{+}\cdot\underset{\sim}{k}) \ . \tag{15}$$

Now, the form (14) is clearly self adjoint, and the characteristic polynomial (15) is not substantially different from that for the first set of potentials (11). We note that the self-adjoint property of the system does not depend on any assumption about $\underset{\sim}{\varepsilon}$. The system (12),(13) is the set of Euler equations of the variational form

$$I = \int (dV)\ \{-\underset{\sim}{\nabla} \times \underset{\sim}{A}^*\cdot\underset{\sim}{\nabla} \times \underset{\sim}{A} + A^*\cdot\underset{\sim}{\varepsilon}\cdot\underset{\sim}{A} + i\phi\underset{\sim}{\nabla}\cdot(\underset{\sim}{A}^*\cdot\underset{\sim}{\varepsilon}) - i\phi^*\underset{\sim}{\nabla}\cdot(\underset{\sim}{\varepsilon}^{+}\cdot\underset{\sim}{A})\} \ . \tag{16}$$

If $\varepsilon^+ = \varepsilon$ then the full variational form (16) and not just the differential operator is self-adjoint. Such a variational form should lead to good numerical solution schemes by finite element or other finite difference approximations when appropriate boundary conditions and other subsidiary relations are given. Consistent with the variational form it should be possible to specify ϕ_2 and the tangential component of $\underset{\sim}{E}$ on a boundary. Although $\underset{\sim}{E}$ considered as a solution of the basic differential equation (4) is also the variational solution of (16) with $\phi = \phi^* \equiv 0$ when $\underset{\sim}{E}$ is identified with $\underset{\sim}{A}$, this latter form is degenerate and has given rise to serious numerical difficulties.[5] We expect that (16) as formulated should not give rise to such problems.

We expect that a time harmonic wave propagation problem or a scattering problem should give rise to an elliptic set of differential equations, as happens in electromagnetic theory or quantum mechanics. We shall see that this typical situation fails for the cold plasma model. The necessary and sufficient condition for ellipticity is that

the characteristic form (11) or (15) have no zeroes for any real, non-zero, vector $\underset{\sim}{k}$. The condition reduces to the assertion that

$$C = \underset{\sim}{k} \cdot \underset{\sim}{\varepsilon} \cdot \underset{\sim}{k} \tag{17}$$

never vanish for real, non-zero, $\underset{\sim}{k}$. If we rotate axes so that the equilibrium magnetic field lies in the z direction we find that

$$C = (k_x^2+k_y^2)\left[1 - \sum_{(j)} \omega_{p_j}^2/(\omega^2-\Omega_{c_j}^2)\right] + k_z^2\left[1 - \left(\sum_{(j)} \omega_{p_j}^2\right)/\omega^2\right] ,$$

or the necessary and sufficient condition for ellipticity is

$$\left[1 - \sum_{(j)} \omega_{p_j}^2/(\omega^2-\Omega_j^2)\right]\cdot\left[1 - \left(\sum_j \omega_{p_j}^2\right)/\omega^2\right] > 0 . \tag{18}$$

When the condition (18) fails the system (8), (9) has two real (hyperbolic) characteristics and the system (12),(13) has four real characteristics. For frequencies much less than the plasma frequency of the medium, $\sum_{(j)} \omega_{p_j}^2$, the condition for ellipticity is

$$1 < \sum \omega_{p_j}^2/(\omega^2-\Omega_j^2) . \tag{19}$$

More generally, the first factor in (18) changes sign when inequality in (19) is replaced by equality, the condition for what are usually called the lower and upper hybrid resonances. Thus, the lower and upper hybrid resonances may be interpreted as the occurrence of a transition from an elliptic to a hyperbolic system. Another transition occurs at the cyclotron resonances where $\omega^2 = \Omega_{c_j}^2$. It is clear that the cold plasma model fails near these values as $\underset{\sim}{\sigma}$ becomes unbounded.

Finally the second factor in (18) changes sign at the plasma resonance $\omega^2 = \sum_{(j)} \omega_{p_j}^2$. The meaning of the term resonance is quite different when we refer to cyclotron resonances at which the cold plasma model fails or to hybrid and plasma resonances where the cold plasma model appears to continue to be satisfactory. In the latter cases the infinite homogeneous dispersion relation, slab models, or geometrical optics indicate the presence of "resonance" phenomena. It is our purpose to explore further in subsequent sections the cyclotron and hybrid resonances and their significance and relation to possible energy absorption in a plasma.

A transition from ellipticity to hyperbolicity is unusual in mathematical physics, but it does occur in fluid dynamics and in plasma physics, for instance. The existence of such a transition may give rise to singular differential equations and requires that we decide how to select solutions from a potentially singular set. Steady, compressible fluid flow in the transonic regime generates a transition from elliptic to hyperbolic flow across the sonic surface.(6) The flows chosen are smooth across the sonic surface and satisfy a causality condition. For transonic flows the causality condition is equivalent to the specification of data on the real, forward-facing characteristics, no data on the backward-facing characteristics, and typical data in the elliptic part of the region. In the guiding center plasma of guiding center fluid there is also a transition from an elliptic system to a hyperbolic system.(7) In this case, the transition is indicative of instabilities not resolved by the models. Here, a sonic transition indicates the failure of a model and an incomplete physical representation of the system.

Yet another method of selecting solutions of wave propagation problems is to require that the solution be causal in another sense. Causality requires that the solution $[\underset{\sim}{E}(\underset{\sim}{x},\omega),\underset{\sim}{B}(\underset{\sim}{x},\omega)]$ possess an analytic continuation in the complex ω plane, $\mathrm{Im}\,\omega < 0$ and this continuation must be bounded as $\mathrm{Im}\,\omega \to -\infty$. Causality in this sense follows from the observation that the time dependent problem with solution $[\underset{\sim}{E}(\underset{\sim}{x},t),\underset{\sim}{B}(\underset{\sim}{x},t)]$ must possess a well-behaved Laplace transform with Laplace transform variable $p = i\omega$, and that the time harmonic dependence solution is the limit of the Laplace transform solution as $\mathrm{Re}\,p = -\mathrm{Im}\,\omega$ tends to zero. The time dependent solution is a manifestly well-behaved system if one recalls that the cold plasma conductivity comes from the linearized momentum equation for each species

$$m_j n_o(\underset{\sim}{r}) \frac{\partial \underset{\sim}{u}_j}{\partial t} = e_j(\underset{\sim}{E} + \underset{\sim}{u}_j \times \underset{\sim}{B}^{(0)})$$

and the current which is to be inserted in Maxwell's equations

$$\underset{\sim}{J} = \sum_{(j)} e_j n_o(\underset{\sim}{r})\underset{\sim}{u}_j \ .$$

This system and Maxwell's equations are exponentially bounded, so that a Laplace transform must exist. With this interpretation $\underset{\sim}{E}(\underset{\sim}{x},\omega)$ and $\underset{\sim}{B}(\underset{\sim}{x},\omega)$ may be singular in $\underset{\sim}{r}$ when ω is real. This approach has been applied in many cases to plasma wave propagation problems.[8] In particular with another author we have applied it to lower hybrid propagation for a perpendicularly stratified medium,[2] similar to many older analogous treatments. We believe that the problem of wave

propagation in a cold plasma employs each of these methods of selecting a physically relevant solution and has a few additional unique questions yet to be resolved.

III. WAVE PROPAGATION IN AXISYMMETRIC GEOMETRY

In the previous section we studied the cold plasma model in the general case. We showed that the wave propagation problem reduced to a sixth order set of differential equations for the electromagnetic potentials $\underset{\sim}{A}$ and ϕ. For the remainder of the paper we turn to the experimentally significant, but special, case of wave propagation for a plasma equilibrium which is axisymmetric. We show that with a different choice of potentials the governing system of equations is of fourth order, and we show that these equations are also derivable from a variational principle different from (17) when $\underset{\sim}{\varepsilon}$ is hermitian. We examine the type of the system of equations and compare the elliptic to hyperbolic transition in this case to the previous case. For application in subsequent sections, we also present simple explicit representations of $\underset{\sim}{\varepsilon}$ and various pertinent expressions for an axisymmetric equilibrium. We employ the usual cylindrical coordinates (r,θ,z), and orthonormal unit vectors $(\hat{r},\hat{\theta},\hat{z})$. When $\underset{\sim}{\varepsilon}$ is referred to the basis set $(\hat{r},\hat{\theta},\hat{z})$ then by axisymmetry the elements $\varepsilon_{rr}, \varepsilon_{r\theta}, \varepsilon_{rz}$, etc., are all functions of r and z alone. We may then assume that the time varying electromagnetic fields have the θ dependence $e^{im\theta}$, and we treat separately the case $m = 0$ and $m \neq 0$.

A. The Case $m = 0$

If we take E_θ and B_θ as the two basic functions, or potentials, then the $\hat{r}$ and $\hat{z}$ components of (1) and (2) yield

$$B_r = -iE_{\theta,z} \tag{20a}$$

$$B_z = i(rE_\theta)_{,r}/r \tag{20b}$$

$$D_r = iB_{\theta},r \tag{21a}$$

$$D_z = -i(rB_{\theta}),z/r \ . \tag{21b}$$

We may solve (21) for E_r and E_z in terms of E_θ and B_θ to find

$$\Delta E_r = i[\varepsilon_{rz}(rB_{\theta}),_r/r + \varepsilon_{zz}\ B_{\theta},z] + (\varepsilon_{rz}\varepsilon_{z\theta} - \varepsilon_{r\theta}\varepsilon_{zz})E_\theta \tag{22a}$$

$$\Delta E_z = -i[\varepsilon_{rr}(rB_{\theta}),_r/r + \varepsilon_{zr}\ B_{\theta},z] + (\varepsilon_{zr}\varepsilon_{r\theta} - \varepsilon_{z\theta}\varepsilon_{rr})E_\theta \ , \tag{22b}$$

where

$$\Delta = \varepsilon_{rr}\varepsilon_{zz} - \varepsilon_{rz}\varepsilon_{zr} \ . \tag{23}$$

The $\hat{\theta}$ components of (1),(2) are as yet unsatisfied and they imply

$$E_{r,z} - E_{z,r} = -iB_\theta \tag{24a}$$

$$B_{r,z} - B_{z,r} = iD_\theta = i(\varepsilon_{\theta r}E_r + \varepsilon_{\theta\theta}E_\theta + \varepsilon_{\theta z}E_z) \ . \tag{24b}$$

Thus, (24) together with the definitions (20),(22) and (23) constitutes two second order equations for E_θ and B_θ. It is easy to construct the characteristic form C of the system and we find

$$C = \underset{\sim}{k}^2\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}\cdot\underset{\sim}{k} \tag{25}$$

where $\underset{\sim}{k} = (k_r,0,k_z)$. Clearly (25) is a reduction of the previously given forms (11) or (15) to the case with $k_\theta = 0$ corresponding to the hypothesis of no θ variation of the several functions. The forms that characterize ellipticity, (18) and (19), no longer apply as we

can no longer rotate vectors at will as was required to obtain these forms. Rather than write out at this point the explicit equations corresponding to (24) with the substitutions (20), (22) and (23), we give a variational form whose Euler equations generate this sytem,

$$
\begin{aligned}
I = \int rdrdz \, \{ & [\underset{\sim}{\nabla}(rE_\theta^*)\cdot\underset{\sim}{\nabla}(rE_\theta)]/r^2 + [\underset{\sim}{\nabla}(rB_\theta^*)\cdot\underset{\sim}{\varepsilon}\cdot\underset{\sim}{\nabla}(rB_\theta)]/(r^2\Delta) \\
& + iE_\theta[(rB_\theta^*)_{,r}(\varepsilon_{zr}\varepsilon_{r\theta} - \varepsilon_{z\theta}\varepsilon_{rr})/r + B_{\theta,z}^*(\varepsilon_{r\theta}\varepsilon_{zz} - \varepsilon_{rz}\varepsilon_{z\theta})]/\Delta \\
& - iE_\theta^*[(rB_\theta)_{,r}(\varepsilon_{rz}\varepsilon_{\theta r} - \varepsilon_{\theta z}\varepsilon_{rr})/r + B_{\theta,z}(\varepsilon_{\theta r}\varepsilon_{zz} - \varepsilon_{zr}\varepsilon_{\theta z})]/\Delta \\
& - B_\theta^* B_\theta - E_\theta^* E_\theta [\det(\underset{\sim}{\varepsilon})]/\Delta \}
\end{aligned}
\tag{26}
$$

In the form (26) $\underset{\sim}{\nabla}\psi = \hat{r}\psi_{,r} + \hat{z}\psi_{,z}$. If $\underset{\sim}{\varepsilon}$ is hermitian then (26) is self-adjoint. Such variational forms frequently form the basis for efficient computational algorithms.

B. <u>The Case</u> $m \neq 0$

The general case is much more difficult to treat. We start by considering the prototype equation

$$\underset{\sim}{\nabla}\times\underset{\sim}{L} = i\underset{\sim}{M} \; . \tag{27}$$

Now, $\underset{\sim}{L}$ and $\underset{\sim}{M}$ satisfying (27) exist only if

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{M} = 0 \; . \tag{28}$$

Hence, we invert the role of the equations (27),(28) and we assume that (28) holds and ask for solutions to (27). It is easy to see that if $\underset{\sim}{M}$ satisfies (28) then a particular solution of (27) is

$$\underset{\sim}{L}' = (-rM_z/m, 0, rM_r/m)$$

so that the most general solution of (27), given (28) is

$$\underset{\sim}{L} = (-rM_z/m, 0, rM_r/m) + \underset{\sim}{\nabla}\chi \ . \tag{29}$$

Hence, the equation (27) is replaced by (28) and the representation (29). We now apply this approach to our original system (1),(2) and we have the two differential equations

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \tag{30}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{D} = 0 \tag{31}$$

and the representations

$$\underset{\sim}{E} = (rB_z/m, 0, -rB_r/m) + \underset{\sim}{\nabla}\lambda \tag{32}$$

$$\underset{\sim}{B} = (-rD_z/m, 0, rD_r/m) + \underset{\sim}{\nabla}\mu \ . \tag{33}$$

The $\hat{\theta}$ components of (32) and (33) are

$$E_\theta = im\lambda/r \tag{34}$$

$$B_\theta = im\mu/r \ , \tag{35}$$

so that the potentials λ and μ are essentially E_θ and B_θ. Thus, whatever value m has, the basic potentials are E_θ and B_θ.

The $\hat{r}$ and $\hat{z}$ components of (32) and (33) are equivalent to the system for D_r, D_z, B_r and B_z

$$D_r - (r/m)(\varepsilon_{rr}B_z - \varepsilon_{rz}B_r) = i\varepsilon_{r\theta}m\lambda/r + \varepsilon_{rr}\lambda_{,r} + \varepsilon_{rz}\lambda_{,z}$$

$$D_z - (r/m)(\varepsilon_{zr}B_z - \varepsilon_{zz}B_r) = i\varepsilon_{z\theta}m\lambda/r + \varepsilon_{zr}\lambda_{,r} + \varepsilon_{zz}\lambda_{,z} \quad (36)$$

$$B_r + (r/m)D_z = \mu_{,r}$$

$$B_z - (r/m)D_r = \mu_{,z}$$

We may easily solve (36) for D_r, D_z, B_r, and B_z in terms of λ and μ to find

$$\bar{\Delta}D_r = i(m/r)[\varepsilon_{r\theta} + (r/m)^2(\varepsilon_{z\theta}\varepsilon_{rz} - \varepsilon_{r\theta}\varepsilon_{zz})]\lambda + [\varepsilon_{rr} - (r/m)^2\Delta]\,[\lambda_{,r} + (r/m)\mu_{,z}]$$
$$+ \varepsilon_{rz}[\lambda_{,z} - (r/m)\mu_{,r}]$$

$$\bar{\Delta}D_z = i(m/r)[\varepsilon_{z\theta} + (r/m)^2(\varepsilon_{r\theta}\varepsilon_{zr} - \varepsilon_{z\theta}\varepsilon_{rr})]\lambda + \varepsilon_{zr}[\lambda_{,r} + (r/m)\mu_{,z}]$$
$$+ [\varepsilon_{zz} - (r/m)^2\Delta][\lambda_{,z} - (r/m)\mu_{,r}]$$

$$\bar{\Delta}B_r = -i[\varepsilon_{z\theta} + (r/m)^2(\varepsilon_{r\theta}\varepsilon_{zr} - \varepsilon_{z\theta}\varepsilon_{rr})]\lambda - (r/m)\varepsilon_{zr}[\lambda_{,r} + (r/m)\mu_{,z}]$$
$$- (r/m)[\varepsilon_{zz} - (r/m)^2\Delta]\lambda_{,z} + [1 - (r/m)^2\varepsilon_{zz}]\mu_{,r} \quad (37)$$

$$\bar{\Delta}B_z = i[\varepsilon_{r\theta} + (r/m)^2(\varepsilon_{z\theta}\varepsilon_{rz} - \varepsilon_{r\theta}\varepsilon_{zz})]\lambda + (r/m)[\varepsilon_{rr} - (r/m)^2\Delta]\lambda_{,r}$$
$$+ (r/m)\,\varepsilon_{rz}[\lambda_{,z} - (r/m)\mu_{,r}] + [1 - (r/m)^2\varepsilon_{rr}]\mu_{,z}\ ,$$

where

$$\bar{\Delta} = 1 - (r/m)^2(\varepsilon_{rr}+\varepsilon_{zz}) + (r/m)^4\Delta\ . \quad (38)$$

To complete the representation of the fields in terms of the potentials we need only give D_θ, but $D_\theta = \varepsilon_{\theta r}E_r + \varepsilon_{\theta\theta}E_\theta + \varepsilon_{\theta z}E_z$ so that with (32), (34) and (37) we find

$$\begin{aligned}\Delta D_\theta = i(m/r)\{&\varepsilon_{\theta\theta} + (r/m)^2[\varepsilon_{\theta\theta}(\varepsilon_{rr}+\varepsilon_{zz}) - \varepsilon_{\theta r}\varepsilon_{r\theta} - \varepsilon_{\theta z}\varepsilon_{z\theta}] + (r/m)^4 \det \underset{\sim}{\varepsilon}\}\lambda \\ &+ [\varepsilon_{\theta r} + (r/m)^2(\varepsilon_{\theta z}\varepsilon_{zr}-\varepsilon_{\theta r}\varepsilon_{zz})]\,[\lambda_{,r} + (r/m)\mu_{,z}] \\ &+ [\varepsilon_{\theta z} + (r/m)^2(\varepsilon_{\theta r}\varepsilon_{rz}-\varepsilon_{\theta z}\varepsilon_{rr})]\,[\lambda_{,z} - (r/m)\mu_{,r}] \ . \qquad (39)\end{aligned}$$

We now see that $\underset{\sim}{D}$ and $\underset{\sim}{B}$ are represented by the potentials by (35), (37) and (39), so that the governing differential equations are (30) and (31). It is easy to show that the system consists of two second order partial differential equations for λ and μ and that the characteristics of the system are given by the same form as the case $m = 0$, (25). It is clear that our equations are Euler equations of the variational form

$$I = -\int rdrdz(\lambda^*\underset{\sim}{\nabla}\cdot\underset{\sim}{D} + \mu^*\underset{\sim}{\nabla}\cdot\underset{\sim}{B}) \ ,$$

or

$$\begin{aligned}I = \int rdrdz[&D_r(r\lambda^*)_{,r}/r - D_\theta\lambda^*(im/r) + D_z\lambda^*{}_{,z} \\ &+ B_r(r\mu^*)_{,r}/r - B_\theta\mu^*(im/r) + D_z\mu^*{}_{,z}] \ . \qquad (40)\end{aligned}$$

It is tedious but straightforward to show that (40), with the substitutions (35), (37), and (39) is self-adjoint if $\underset{\sim}{\varepsilon}$ is hermitian.

C. Explicit Representations of $\underset{\sim}{\varepsilon}$ and Associated Functions

Before we turn to applications of the material just presented, we give the explicit structure of the $\underset{\sim}{\varepsilon}$ matrix and associated functions. We may take the equilibrium magnetic field to have the form

$$\underset{\sim}{B}_o = B_o[\hat{\theta} \cos \alpha + \sin \alpha(\hat{r} \sin \beta + \hat{z} \cos \beta)] , \qquad (41)$$

where α and β are functions of r and z alone. A right-handed orthonormal set of vectors is

$$\hat{v}_1 = \hat{r} \cos \beta - \hat{z} \sin \beta$$

$$\hat{v}_2 = \hat{\theta} \sin \alpha - \cos \alpha(\hat{r} \sin \beta + \hat{z} \cos \beta)$$

$$\hat{v}_3 = \hat{\theta} \cos \alpha + \sin \alpha(\hat{r} \sin \beta + \hat{z} \cos \beta)$$

in terms of which

$$\underset{\sim}{\varepsilon} = \underset{\sim}{I} - (\hat{v}_1\hat{v}_1+\hat{v}_2\hat{v}_2) \sum_{(j)} X_j/(1-Y_j^2) + i(\hat{v}_1\hat{v}_2-\hat{v}_2\hat{v}_1) \sum_{(j)} X_jY_j/(1-Y_j^2) - \hat{v}_3\hat{v}_3 \sum_{(j)} X_j$$

where

$$X_j = \omega_{p_j}^2/\omega^2 \qquad (42)$$

$$Y_j = \Omega_{c_j}/\omega \ . \qquad (43)$$

We have thus obtained the components of the $\underset{\sim}{\varepsilon}$ tensor:

$$\left.\begin{aligned}
\varepsilon_{rr} &= a - b \sin^2\alpha \sin^2\beta \\
\varepsilon_{\theta\theta} &= a - b \cos^2\alpha \\
\varepsilon_{zz} &= a - b \sin^2\alpha \cos^2\beta \\
\varepsilon_{r\theta} &= \varepsilon^*_{\theta r} = -b \sin\alpha \cos\alpha \sin\beta - i\, c \sin\alpha \cos\beta \\
\varepsilon_{\theta z} &= \varepsilon^*_{z\theta} = -b \sin\alpha \cos\alpha \cos\beta - i\, c \sin\alpha \sin\beta \\
\varepsilon_{rz} &= \varepsilon^*_{zr} = -b \sin^2\alpha \sin\beta \cos\beta + i\, c \cos\alpha
\end{aligned}\right\} \quad (44)$$

where

$$\left.\begin{aligned}
a &= 1 + \sum_{(j)} X_j/(Y_j^2-1) \\
b &= \sum_{(j)} X_j[1 + 1/(Y_j^2-1)] \\
&= a + \sum_{(j)} X_j - 1 \\
c &= \sum_{(j)} X_j Y_j/(Y_j^2-1) \ .
\end{aligned}\right\} \quad (45)$$

Other functions of $\underset{\approx}{\varepsilon}$ which we need for the various representations are

$$\left.\begin{aligned}
\Delta &= \varepsilon_{rr}\varepsilon_{zz} - \varepsilon_{rz}\varepsilon_{zr} = a(a - b \sin^2\alpha) - c^2 \cos^2\alpha \\
f_1 &= i(\varepsilon_{z\theta}\varepsilon_{rz} - \varepsilon_{r\theta}\varepsilon_{zz}) = c(b-a) \sin\alpha \cos\beta - i(c^2-ab) \sin\alpha \cos\alpha \sin\beta \\
f_2 &= i(\varepsilon_{r\theta}\varepsilon_{zr} - \varepsilon_{z\theta}\varepsilon_{rr}) = -c(b-a) \sin\alpha \sin\beta - i(c^2-ab) \sin\alpha \cos\alpha \cos\beta \\
f_3 &= \varepsilon_{r\theta}\varepsilon_{\theta r} + \varepsilon_{z\theta}\varepsilon_{\theta z} - \varepsilon_{\theta\theta}(\varepsilon_{rr}+\varepsilon_{zz}) = 2a(b-a) + (c^2-ab) \sin^2\alpha \\
f_4 &= \varepsilon_{rr} + \varepsilon_{zz} = 2a - b \sin^2\alpha \\
\text{and } \det \underset{\approx}{\varepsilon} &= (c^2-a^2)(b-a) \ .
\end{aligned}\right\} \quad (46)$$

The necessary and sufficient condition for ellipticity is that $\underset{\sim}{k}\cdot\underset{\approx}{\varepsilon}\cdot\underset{\sim}{k}$ not change sign or that the form

$$k_r^2(a - b \sin^2 \alpha \sin^2 \beta) - 2k_r k_z\, b \sin^2 \alpha \sin \beta \cos \beta + k_z^2(a - b \sin^2 \alpha \cos^2 \beta)$$

be definite. Equivalently the system is elliptic if and only if

$$a(a - b \sin^2 \alpha) = a[a \cos^2 \alpha - (\sum_{(j)} X_j - 1) \sin^2 \alpha] > 0 \quad (47)$$

When (47) fails the system has two real (hyperbolic) characteristics. The condition for ellipticity in the three dimensional case, (18), is

$$a(a-b) > 0 \ . \quad (48)$$

If (48) holds then it is direct to show that (47) holds, or if the system is elliptic in the three dimensional sense, then it is elliptic in the two-dimensional sense. If (48) fails then (47), may hold or it may fail, that is, two dimensional sense ellipticity does not imply three dimensional sense ellipticity.

As is well-known, the quantities a,b, and c are grossly disparate in magnitude for applied frequencies less than or on the order of the ion plasma frequency. In the next two sections we examine wave propagation at frequencies on the order of the ion cyclotron frequency and on the order of the ion plasma frequency and lower hybrid frequency. We look to see if the system is significantly simplifed in the frequency range in question. We also examine the implications of resonances -- or elliptic to hyperbolic transitions, and other potential singularities of the equations. The basic small parameter used scale other quantities is the mass ratio η.

$$\eta = m_-/m_+ \tag{49}$$

and provided we assume, as is typical of many laboratory plasmas, that

$$\omega_{pe}/\Omega_{ce} = O(1) \tag{50}$$

then the other non-dimensional frequency ratios are

$$\begin{aligned}
\omega_{pe}/\omega_{pi} &= 1/\sqrt{\eta} \\
\omega_{pe}/\Omega_{ci} &= O(1/\eta) \\
\Omega_{ce}/\omega_{pi} &= O(1/\sqrt{\eta}) \\
\Omega_{ce}/\Omega_{ci} &= 1/\eta \\
\omega_{pi}/\Omega_{ci} &= O(1/\sqrt{\eta}) \ .
\end{aligned} \tag{51}$$

We have now laid out the basic material we need for applications.

IV. WAVES AT ION CYCLOTRON FREQUENCIES

In this section we examine the nature of the wave propagation problem at frequencies on the order of the ion cyclotron frequency. We examine the resonances, elliptic-hyperbolic transition, other singularities of the system, possible simplifications of the system, and necessary rescaling of the variables. When we assume that

$$\Omega_{c_i}/\omega = O(1) \tag{52}$$

it follows easily from the definitions and (51) that

$$\begin{aligned} X_i &= O(1/\eta) \\ Y_i &= O(1) \\ X_e &= O(1/\eta^2) \\ Y_\theta &= O(1/\eta) \end{aligned} \tag{53}$$

so that

$$\begin{aligned} a &= O(1/\eta) \\ b &= O(1/\eta^2) \\ c &= O(1/\eta) \end{aligned} \tag{54}$$

The problems in a mirror machine, $\cos\alpha = 0$, and a Tokamak, $\cos\alpha \sim 1$, are very different in structure. Another potential complication in a Tokamak concerns the ratio of the poloidal magnetic field to the total magnetic field, which is $\sin\alpha$ and is small. In the frequency range of investigation even $\sin^2\alpha$, which is on the order of a few precent or more, is much larger than η. Thus, we do not treat $\sin\alpha$ or $\sin^2\alpha$ as a small parameter to be scaled in η. In the next section on lower

hybrid frequencies we will have to scale $\sin\alpha$ with η in order to maintain physical relevance.

We first examine the elliptic-hyperbolic transition and resonance in the given frequency range (52). For a mirror machine $\cos^2\alpha = 0$, and $\sum_{(j)} X_j >> 1$, so that the system is elliptic, see (47), for $a < 0$ or $\omega^2 > \Omega_{c_i}^2$. Thus, above cyclotron frequency the system is elliptic and below that frequency it is hyperbolic. Here, the elliptic hyperbolic transition and cyclotron resonance coincide. For a Tokamak the transition is different. When $a < 0$, (47) clearly holds so that the system is elliptic. When $a > 0$ the inequality (47) requires

$$a \cos^2\alpha > (\sum_{(j)} X_j - 1)\sin^2\alpha \ ,$$

or to leading order in η

$$Y_i^2 - 1 < (m_e/m_i)\cot^2\alpha \ .$$

The elliptic-hyperbolic transition occurs at

$$\Omega_{c_i}/\omega = 1 + (1/2)(m_e/m_i)\cot^2\alpha + \ldots \ , \qquad (55)$$

a frequency slightly different from the cyclotron frequency. In both cases the system is elliptic for high frequencies and hyperbolic at lower frequencies.

We examined the situation for a mirror machine near resonance in another paper (1). In this case the resonance, elliptic-hyperbolic transition, and breakdown of the cold plasma model all coincide. There we started with the Vlasov equation and used a boundary layer expansion of the system near resonance. We showed that the cold plasma model could still be used provided we imposed appropriate jump conditions

across the resonance. On either side of the resonance the electric field had the structure

$$\underset{\sim}{E} = \underset{\sim}{E}^{(o)}(y) + x \log |x| \, \underset{\sim}{E}^{(1)}(y) + x \, \underset{\sim}{E}^{(2)}(y) + \ldots , \qquad (56)$$

where x is a coordinate orthogonal to the resonant surface and y is a coordinate in the surface. The electric field also satisfied the jump conditions across the resonance surface

$$[\underset{\sim}{E}] = 0$$

$$[(\hat{n}\cdot\underset{\sim}{\nabla})\underset{\sim}{E}] = \underset{\sim}{\underset{\sim}{S}}\cdot\underset{\sim}{E} \quad , \qquad (57)$$

where $\underset{\sim}{\underset{\sim}{S}}$ is a specified matrix. More importantly, we could obtain the relations (56),(57) solely within the cold plasma model by the imposition of one form of causality condition on the solution. We recall that for causality to hold in this case the solution is obtained by the limiting procedure $\mathrm{Im}\,\omega \to 0$, $\mathrm{Im}\,\omega < 0$.

In a Tokamak there is an elliptic hyperbolic transition (55) and the possibility of a nearby cyclotron resonance. We note that for a very low beta Tokamak $\cot^2 \alpha$ can be quite large, so that the transition may be well away from cyclotron resonance. Further, if one uses the variational forms for the wave propagation equations together with the many defintions and explicit forms (45) and (46) it is direct to show that the forms (26) and (40) are well-behaved as $\omega \to \pm\Omega_c$. Thus, the <u>only</u> singularity of the system is the elliptic-hyperbolic transition, which typically is very close to cyclotron resonance. Even though the only singularity occurs when (55) is satisfied we know that the cold plasma model fails for $\omega^2 = \Omega^2_{c_i}$. Except in a <u>very</u> low beta Tokamak we expect the transition (55) to occur in a boundary layer region in which the Vlasov equation must be employed, exactly as we did for a mirror

machine. We believe that if such an analysis is carried out answers very similar to (57) would be found. We do not present such results here.

There is one other pair of possible singularities in the differential equations. If we examine (22) for the case $m = 0$ and (37) for the case $m \neq 0$, it is clear the the representations are singular for $\Delta = 0$ when $m = 0$ and $\bar{\Delta} = 0$ for $m \neq 0$. Both singularities occur when one solves for the fields in terms of the potentials. We believe that the singularities are only apparent and do not show up in the solutions in a significant manner. In the full, three dimensional representation of the solutions neither $\Delta = 0$ or $\bar{\Delta} = 0$ gave rise to any singular behavior of the solutions. It was only with the particular representations in terms of potentials that any singularities occur. Thus, we expect that the solutions for the potentials will be such that $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are well-behaved near $\Delta = 0$ or $\bar{\Delta} = 0$. We expect the solutions for E_θ and B_θ to be regular near those surfaces and such that $\underset{\sim}{E}$ and $\underset{\sim}{B}$ defined by (20),(22) or (32),(33), and (37) may be defined in the limit as Δ or $\bar{\Delta}$ tends to zero. Further examination of the effects of these apparent singularities may be carried out by the methods of the following section.

We may finally consider whether the scalings (52), (53) and (54) yield a significantly simpler set of equations than (22), (23), (24) or (30), (31), (35), (37), (38), (39). We recall that we assumed a distance scale length $c/\omega = O(c/\Omega_{c_i})$. For the waves at these frequencies the correct scale length is c/ω_{p_i}, a much smaller unit. Thus if we introduce

$$\bar{r} = r\,\Omega_{c_i}/\omega_{p_i}$$

$$\bar{z} = z\,\Omega_{c_i}/\omega_{p_i}\,, \tag{58}$$

then we expect derivatives with respect to $\bar{r}$ and $\bar{z}$ to be $O(1)$. It is simpler, and consistent with (58) to introduce

$$r' = \sqrt{\eta}\, r$$

$$z' = \sqrt{\eta}\, z \tag{59}$$

and again derivatives with respect to r' and z' are $O(1)$ in η. It is now tedious, but straightforward to introduce the scalings (52), (53), (54) and (59) into the governing differential equations and provided we take

$$E_\theta = O(1)$$

$$B_\theta = O(\eta^{-1/2} \tag{60}$$

when $m = 0$ and

$$\lambda = O(1)$$

$$\mu = O(\eta^{-1/2}) \tag{61}$$

when $m \neq 0$, we see that all the coefficients are essentially of the same order of magnitude. Thus, the scalings leave two second order differential equations with the same structure. The coefficients may simplify somewhat, but no significant changes occur.

We have carried out our study of ion cyclotron waves without explicitly imposing the condition that the component of $\underset{\sim}{E}$ parallel to the equilibrium magnetic field must be small. It is interesting to see how this property of the solution appears in our treatment. A direct evaluation of $E_\parallel$ from (22) or (32) and (37) in terms of the appropriate potentials shows that $E_\parallel$ is small, whatever the potentials are, provided (60) or (61) hold. Hence, we need not consider $E_\parallel$ further.

The scalings (52), (53), (54) together with (59) move the singularities $\Delta = 0$ or $\bar{\Delta} = 0$ even further out of the range of practical interest, so there is little reason to study them in detal except very near ion cyclotron resonance. Near cyclotron resonance more delicate analysis based on the Vlasov equation is needed.

Finally we observe that the numerical calculation of wave propagation in a Tokamak or mirror machine remains a difficult problem unlike other wave propagation problems. Even after the introduction of the scalings (60) and (61) and the jump conditions (57) appropriate to a mirror machine or Tokamak, we are left with differential equations which are partly elliptic and in other regimes partly hyperbolic. Although there is some experience with such problems in transonic flow, there is little known about the actual problems we have. Presumably the variational forms (26) and (40) would be the basis of such calculations.

V. WAVES AT LOWER HYBRID FREQUENCIES

In this section we turn to wave propagation problems at frequencies on the order of the lower hybrid -- or ion plasma -- frequency. We obtain the scaled equations and we study the nature of the solutions near the elliptic-hyperbolic transition, corresponding to the lower hybrid resonance. We concentrate the treatment on the case of $m = 0$ modes in a Tokamak, although the methods apply quite generally. The formulas are much simpler in this case. We are interested in the properties of the solution when we assume

$$\omega_{p_i}/\omega = O(1) \tag{62}$$

so that

$$\begin{aligned} X_i &= O(1) \\ Y_i &= O(\sqrt{\eta}) \\ X_e &= O(1/\eta) \\ Y_e &= O(1/\sqrt{\eta}) \end{aligned} \tag{63}$$

and

$$\begin{aligned} a &= O(1) \\ b &= O(1/\eta) \\ c &= O(1/\sqrt{\eta}) \quad . \end{aligned} \tag{64}$$

The condition for ellipticity of the system, (47) reduces to $a < 0$, or

$$1 + \omega_{p_e}^2/\Omega_{c_e}^2 < \omega_{p_i}^2/\omega^2 \ , \tag{65}$$

which we may write as

$$1/\omega^2 > 1/\omega_{p_i}^2 + 1/|\Omega_{c_i}\Omega_{c_e}| \ . \tag{66}$$

Equality in (65) or (66) is just the condition for the lower hybrid resonance, so that here the elliptic-hyperbolic transition is just the lower hybrid resonance. For many equilibria $\Omega_{c_i}\Omega_{c_e}$ changes much less than $\omega_{p_i}^2$, so that if the transition occurs, the system is hyperbolic in the lower density, outer region and elliptic in the higher density, inner region. This arrangement is not at all typical for wave propagation problems and may well affect accessibility and antenna design.

Unlike the frequency range near the ion cyclotron resonance, where one might argue that the cold plasma approximation is inappropriate, near the lower hybrid resonance there is no obvious argument that the cold plasma model is poor. Thus, we believe that we should be able to study waves in this frequency domain within the cold plasma model merely by the imposition of an appropriate causality condition. For simplicity we treat only the case m = 0, and ultimately only for a Tokamak. There are several complications that we must address, but first we give the explicit equations derivable from (26)

$$[(rE_{\theta},_r/r],_r + E_{\theta},_{zz} + E_{\theta}(\det \underset{\sim}{\varepsilon})/\Delta = [f_2^*(rB_{\theta}),_r - f_1^* B_{\theta},_z]/\Delta \tag{67}$$

$$\begin{aligned}&[(\varepsilon_{rr}/\Delta)(rB_{\theta}),_r/r + (\varepsilon_{zr}/\Delta)B_{\theta},_z],_r + [(\varepsilon_{rz}/\Delta)(rB_{\theta}),_r/r \\ &\quad + (\varepsilon_{zz}/\Delta)B_{\theta},_z],_z + B_{\theta} \\ &= [-f_2(rE_{\theta}),_r/r + f_1 E_{\theta},_z]/\Delta \ .\end{aligned} \tag{68}$$

If we now introduce a vector in the direction of the poloidal magnetic field

$$\hat{u} = \sin\beta\ \hat{r} + \cos\beta\ \hat{z} \tag{69}$$

and an orthogonal vector in the poloidal plane

$$\hat{v} = +\cos\beta\ \hat{r} - \sin\beta\ \hat{z} \tag{70}$$

so that $\hat{v}, \hat{\theta}, \hat{u}$ form a right-handed orthogonal set, then we may rewrite the system as

$$[(rE_{\theta,r})/r]_{,r} + E_{\theta,zz} + E_\theta(\det \underset{\sim}{\varepsilon})/\Delta = \{-c(b-a)\sin\alpha\ [\hat{u}\cdot\underset{\sim}{\nabla}(rB_\theta)]/r$$
$$+ i(c^2-ab)\sin\alpha\cos\alpha\ [\hat{v}\cdot\underset{\sim}{\nabla}(rB_\theta)]/r\}/\Delta \tag{71}$$

$$r\underset{\sim}{\nabla}\cdot\{[a/(r^2\Delta)]\underset{\sim}{\nabla}(rB_\theta)\} - r\underset{\sim}{\nabla}\cdot\{[(b\sin^2\alpha)/(r^2\Delta)]\ \hat{u}(\hat{u}\cdot\underset{\sim}{\nabla})(rB_\theta)\}$$
$$- i\hat{\theta}\cdot\underset{\sim}{\nabla}(rB_\theta)\times\underset{\sim}{\nabla}\{(c\cos\alpha)/(r\Delta)\} + B_\theta$$
$$= \{c(b-a)\sin\alpha[\underset{\sim}{\hat{u}}\cdot\underset{\sim}{\nabla}(rE_\theta)]/r + i(c^2-ab)\sin\alpha\cos\alpha[\underset{\sim}{\hat{v}}\cdot\underset{\sim}{\nabla}(rE_\theta)]/r\}/\Delta \tag{72}$$

We see easily that the second order differential operator in (71) is essentially the Laplacian, while in (72) the first of the two second order operators is also the Laplacian while the second of the two is essentially $(\underset{\sim}{\hat{u}}\cdot\underset{\sim}{\nabla})^2$. The first equation is always elliptic while the second equation, since $b \gg |a| > 0$ is elliptic for $a < 0$ and hyperbolic otherwise.

With the equations displayed explicitly, we wish to examine two further questions. We wish to determine the scaling of the solutions in η that generates reasonable electromagnetic fields and we wish to

study the solutions near the elliptic-hyperbolic transition -- or lower hybrid resonance. We must distinguish carefully the case of a Tokamak, $\cos\alpha \sim 1$, and the case of a mirror, $\sin\alpha = 1$. It is clear that different terms of different orders of magnitude in η appear in the two cases. We choose to examine the case of a Tokamak, the mirror could be done similarly. In the previous section, when we treated the Tokamak we took $\sin^2\alpha = B_p^2/B^2$ to be of order one, but here the relative magnitudes of the terms in (71) and (72) are not so different that we can afford to ignore the smallness of $\sin^2\alpha$. Hence, consistent with situations of physical interest we take

$$\sin\alpha = O(\eta^{1/4}) \ . \tag{73}$$

We now turn to the scaling of the solutions in η. It is easy to see that if we select a scaling of the electromagnetic wave with

$$\begin{aligned} E_\theta &= O(1) \\ B_\theta &= O(\eta^{-1/2}) \\ \nabla_\parallel &= O(\eta^{-1/4}) \\ \nabla_\perp &= O(\eta^{-1/2}) \end{aligned} \tag{74}$$

then we have a self-consistent system with essentially all the terms of (71), (72) included and of the same order of magnitude. We do not modify the characteristic equilibrium scaling length from c/ω_{p_i}.

It is possible to see in (71), (72) with the scalings (73), (74) a significant failure of conventional geometrical optics previously identified in slab geometry.[2] There is a term in (72) proportional to $\underset{\sim}{\nabla}[(c\cos\theta)/(r\Delta)]$ even though we have assumed that the equilibrium varies on the distance scale c/ω_{p_i} while the waves vary on the faster

distance scale $\eta^{-1/4}c/\omega_{p_i}$ or $\eta^{-1/2}c/\omega_{p_i}$. Thus, the condition that the waves vary more rapidly than the equilibrium is not adequate to justify the use of geometrical optics on the original system (1), (2). The result should not be surprising since the elements of $\underset{\sim}{\sigma}$ in the equation depend non-trivially on the geometrical optics parameter. Thus, cut-off and propagation properties of lower hybrid waves derivable from the infinite homogeneous medium are incorrect as they miss such terms. We have already shown in slab geometry a major modification in propagation characteristics that originates with such a term. We could, however, use (71), (72) with the scalings (73), (74) as the basis for a geometrical optics treatment of lower hybrid waves, and this analysis would contain the effects of spatial gradients. That is, we could take the system (71), (72) and assume that (rE_θ) and (rB_θ) vary rapidly in r and z while all equilibrium quantities and their derivatives vary of order one in r and z. This system would then be the basis for the usual geometrical optics procedure. The introduction of the scaling (74) and coordinates parallel and perpendicular to $\underset{\sim}{B}$ before the geometrical optics approximation would simplify the process considerably.

The properties we have just given apply equally well for the modes with $m \neq 0$. One can show that the system of equations (30), (31) with the substitutions (35), (37), (39) also distinguishes parallel and perpendicular derivatives of the waves and leads to scalings of the form (73), (74) and to the non-trivial appearance of spatial gradients of equilibrium quantities. We do not pursue this question or wave propagation in a mirror further here.

The last question we examine is the nature of the solution near the elliptic-hyperbolic transition. We start from the system (71), (72). Since very little is known in general about systems of equations which change type, we recognize that many of our assertions are in fact

conjectures. We base our comments on the information available to us. We first transform the equations to normal form for a second order system. We find that there are two fundamentally different situations. When the resonance surfaces, $a = 0$, coincide with flux surfaces the plasma behaves like a perpendicular stratified medium and there is energy absorption along the entire resonance -- or flux -- surface. The results in this case could be obtained by rather standard inner-outer boundary layer analysis and are rather compelling. The situation when the resonance surfaces do not coincide with flux surfaces is quite different. In the neighborhood of these points where the two surfaces are not tangent the normal form of the equation (72) is essentially the Tricomi equation of transonic flow.[3] In the neighborhood of such points the likely interpretation of the condition of causality is that data is given on some characteristics and not on others, but no singularity of the solution -- or energy absorption -- is expected. Recall that the local Ohmic energy production rate is $\mathrm{Re}\,(\underset{\sim}{E}^{*}\cdot\underset{\sim}{J})$, which vanishes identically for the cold plasma conductivity. Only if $\underset{\sim}{E}$ is singular is heating possible. Near the points of tangency of the resonance surfaces and flux surfaces it is not possible to transform the solution smoothly to the Tricomi form. We believe that energy absorption is possible near these points, but we are unable to provide a definite prescription for its computation beyond the conventional procedure of giving ω a small imaginary part and computing the solution. If we examine a simple model problem that corresponds to the case of tangency of the two surfaces we can show that the family of special solutions corresponding to similarity solutions has no singularities on the resonance surface, but may be singular at the point of tangency of the two surfaces. Finally, an examination of the results of a geometrical optics solution[9] of the system also indicates that at points at which the resonance and flux surfaces are not tangent

no peculiarity of the solutions appears while at points of tangency a resonance behavior does occur. Thus, we believe that energy absorption occurs only near the points of tangency of resonance and flux surfaces.

As we noted previously, of the two equations (71),(72), the first is always elliptic while the second exhibits the change of type. Thus, it is plausible, and greatly simpler, to consider that (71) determines rE_θ when rB_θ is known, while (72) gives rB_θ with rE_θ specified. We consider rE_θ known and concentrate on (72) for rB_θ alone. Further, since we expect the highest order derivatives to be dominant, it should suffice to consider the form

$$F = a(r,z)(u,_{rr}+u,_{zz}) + b \sin^2 \alpha \, (\sin^2 \alpha \, u,_{rr} + 2 \sin \alpha \cos \alpha \, u,_{rz} + \cos^2 \alpha \, u,_{zz}) \, , \tag{75}$$

which gives the leading order terms in (72). We are interested in the structure of (75) near a = 0. If we introduce new coordinates

$$\xi = a(r,z)$$
$$\eta = \eta(r,z) \, , \tag{76}$$

where $\eta(r,z)$ is unspecified then

$$\begin{aligned} F = {} & [a\xi^2,_r + b \sin^2 \alpha(\sin \alpha \, \xi,_r + \cos \alpha \, \xi,_z)^2]u,_{\xi\xi} \\ & + 2[a(\xi,_r\eta,_r+\xi,_z\eta,_z) + b \sin^2 \alpha \, (\sin \alpha \, \xi,_r+\cos \alpha \, \xi,_z)(\sin \alpha \, \eta,_r+\cos \alpha \, \eta,_z))u,_{\xi\eta} \\ & + (a\eta^2,_r + b \sin^2 \alpha \, (\sin \alpha \, \eta,_r+\cos \alpha \, \eta,_z)^2)u,_{\eta\eta} + \text{lower order derivatives} \, . \end{aligned} \tag{77}$$

We may obtain a normal form for F if we eliminate the cross derivatives, or if we select η so that

$$a\underset{\sim}{\nabla}\xi\cdot\underset{\sim}{\nabla}\eta + b\sin^2\alpha(\hat{u}\cdot\underset{\sim}{\nabla}\xi)(\hat{u}\cdot\underset{\sim}{\nabla}\eta) = 0 \tag{78}$$

We must distinguish the cases $\hat{u}\cdot\underset{\sim}{\nabla}\xi = 0$ and $\hat{u}\cdot\underset{\sim}{\nabla}\xi \neq 0$. If $\hat{u}\cdot\underset{\sim}{\nabla}\xi = 0$ then ξ = const. on a flux surface and resonance and flux surfaces coincide. In this case we take η such that $\underset{\sim}{\nabla}\xi\cdot\underset{\sim}{\nabla}\eta = 0$ or the curves η = const. are the orthogonal trajectories to the curves ξ = const. and the flux surfaces. Now,

$$F = L(\xi,\eta)(\xi u_{,\xi\xi} + M(\xi,\eta)u_{,\eta\eta} + \text{lower order derivatives} \tag{79}$$

and $L(\xi,\eta)$ and $M(\xi,\eta)$ are regular and well-behaved near $\xi = 0$, the resonance surface and $M(\xi,\eta) > 0$. When $L(\xi,\eta)$ and $M(\xi,\eta)$ vary slowly with respect to their arguments as they do here it is plausible that the system could be approximated by an ordinary differential equation, and then the system looks like a slab model with a singularity at $\xi = 0$. Slab analysis indicates absorption at $\xi = 0$.[2] We do not pursue this case as it is very untypical of most plasmas.

When $(\hat{\underset{\sim}{u}}\cdot\underset{\sim}{\nabla})\xi \neq 0$, we may use (78) to solve for η, and in this case the normal form of F is

$$F = \tilde{L}(\xi,\eta)\{\tilde{M}(\xi,\eta)u_{,\xi\xi} + \xi u_{,\eta\eta}\} + \text{lower order terms} . \tag{80}$$

We note that (78) and (79) are drastically different equations as the potentially vanishing factor of ξ multiplies different derivatives in each case. Now (80) is essentially the differential equation of transonic flow, or the Tricomi equation.[3] Extensive analysis has indicated that if one gives data on characteristics going into (or out of) the singular surface $\xi = 0$ and data on the boundary on the elliptic side, then there is a unique smooth solution of the system. Such data

corresponds to the data we would give for one of the two equations (71),(72) and it seems quite likely the solution possesses no singularity on the resonance surface. Causality here reduces to the selection of the proper sets of characteristics for data.

We may consider a simple model equation that has many of the properties that interest us. We seek solutions of the equation

$$(x-y^2)u_{,xx} + u_{,yy} = 0 \ . \tag{81}$$

Clearly the equation is elliptic for $x > y^2$ and hyperbolic for $x < y^2$. the resonance surface is $x = y^2$ and the analog of the flux surface is the curve $x = \text{const.}$, on which y varies. Unfortunately we can offer very little information on this equation, but we can study one simple class of solutions, the similarity solutions for which

$$u = x^{\alpha} f(y^2/x) \tag{82}$$

$$(1-\mu)[\alpha(\alpha-1)f(\mu) - 2(\alpha-1)\mu f_{,\mu} + \mu^2 f_{,\mu\mu}] + (2f_{,\mu}+4\mu f_{,\mu\mu}) \ , \tag{83}$$

where

$$\mu = y^2/x \ . \tag{84}$$

We note first that $\mu = 1$, the resonance curve is not a singular point of the equation (83). Thus, we expect no singularities in the family of solutions (82). If, for instance, we sought similarity solutions of the wave equation $u_{,xx} = u_{,yy}$ of the form $u = g(x/y)$ then we would find the light cone $x^2 = y^2$ as a singularity. Returning to (81)-(84) we note that for $|\mu| >> 1$ (or $|x|$ small) $f \sim \mu^{\alpha}$ or $f \sim \mu^{\alpha-1}$ so that u is well-behaved for x small. For our family of similarity solutions there

are no singularities on the resonance surface $\mu = 1$ or flux surface $x = 0$. While this simple analysis is not even conclusive for the simple model equation, it does support our view that the only remaining singularity possible is at the point where the resonance surface and flux surface are tangent $x = y = 0$. At this point singular solutions allow for the possibility of an energy sink for the wave, or a plasma heating zone. Presumably such heating could be calculated by taking $\mathrm{Im}\,\omega \to 0^+$.

Our last approach to this question is based on a geometrical optics approximation solution to the system.(9) Numerical calculation of rays in geometrical optics in a non-perpendicular stratified medium shows that the rays pass through the lower hybrid resonance with no significant modification and that the the ray wave number $\underset{\sim}{k}$ certainly does not become unbounded. This result, found numerically, can be demonstrated analytically as well, if one solves the ray ordinary differential equations near a lower hybrid resonance surface. We do not carry out this elementary exercise in analysis as it is, in fact, essentially the same argument as occurs in going from (77) to (80). The numerical work and elementary analysis both support our contention that except near the points where resonant surfaces and flux surfaces are tangent no singularities or wave heating occur.

The formulation of a wave propagation problem in this frequency range is rather complex. If the wave frequency is everywhere below lower hybrid resonance then the system of equations is elliptic and the problem becomes a standard scattering problem. If the wave frequency is everywhere above lower hybrid resonance then the system of equations is part elliptic and part hyperbolic. Incoming and outgoing wave solutions are defined by the forward and backward facing characteristics so that it should be possible to prescribe incoming waves at one part of the boundary and allow outgoing waves at other

parts of the boundary. When lower hybrid resonance occurs in the plasma, we still have incoming and possibly outgoing waves at part of the boundary, but we also have energy sinks at those few points at which the flux surfaces and resonance surfaces are tangent. A more thorough analysis of the solutions near these points, akin to the study of similarity solutions (81)-(84), is needed to determine the appropriate behavior of the solutions there.

Finally we note that in Tokamaks with $\omega_{p_i}^2 \ll |\Omega_{c_i}\Omega_{c_e}|$, that is for very high field or very low density systems, the resonance surfaces are approximately constant density surfaces and thus likely to be approximately flux surfaces. In this approximately perpendicularly stratified medium we expect strong heating all along the resonance surface exactly as we found in special cases.

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